Satisfying the theorem

Suppose $\alpha \neq 0$

and $p^{\alpha} \mid o(G)$ and $p^{\alpha+1} \nmid o(G)$

$p^{\alpha} \mid o(G) \Rightarrow p \mid o(G)$

$\therefore$ Cauchy theorem for abelian groups

there exist ($\exists$)

$\exists$ an element $a \neq e \in G$ such that $a^p = e$

Let $S = \{ x \in G \mid x^{p^n} = e$ for some integer $n \}$

Claim: S is a subgroup of G $[a^p = (a^p)^1]$

Since $a \neq e$ in G such that $a^p = e$, $a \in S$.

Then S is a non-empty set

[$\therefore S \neq \{e\}$.

Since S is a non-empty finite subset of G]

T.P S is a subgroup of G

It is enough to prove that S is closed. closure

let $x, y \in S$

$x^{p^n} = e$ for some integer n &

$y^{p^m} = e$ for some integer m

$$\text{Now } (xy)^{p^{n+m}} = x^{p^{n+m}} \, y^{p^{n+m}}$$

$$= x^{p^n \cdot p^m} \, y^{p^n \cdot p^m}$$

$$= (x^{p^n})^{p^m} \, (y^{p^m})^{p^n}$$

$$= (e^{p^m}) \, e^{p^n} = e$$

$\therefore xy \in S$.

$\therefore S$ is a subgroup of G

Claim: $o(S) = p^{\beta}$ for some integer β such that $0 < \beta \leq \alpha$

Let q be a prime number such that $q \mid o(S)$,

$q \neq p$

Since S is an abelian group ; By Cauchy Theorem for abelian group, there exist an element $c \neq e \in S$ such that $c^{q} = e$

$q = 3 \quad p^n = 2 \quad n = 2$
$3 < 4 \quad (3,4) = 1$ relatived prime

Since $c \in S$ we have $c^{p^n} = e$ for some integer n

Since p^n and q are relatively prime, $(p^n, q) = 1$ the exist an integer λ & μ such that $\lambda p^n + \mu q = 1$

Now $\quad c = c^1$

$$= c^{\lambda p^n + \mu q}$$

$$= c^{\lambda p^n} \cdot c^{\mu q}$$

$$= (c^{p^n})^{\lambda} \, (c^{q})^{\mu}$$

$$= e^{\lambda} \cdot e^{\mu}$$

$$c = e$$

which is a contradiction $c \neq e$

$p^{\alpha} \mid o(G)$

~~Sin~~ By Lagrange's Theorem, $o(S) \mid o(G)$, we have $o(S) = p^{\beta}$ for some integer β such that $0 < \beta \leq \alpha$

Claim: $\beta = \alpha$

Suppose $\beta < \alpha$

Consider the abelian group G/S

Now $\quad o(G/S) = \dfrac{o(G)}{o(S)}$

$\therefore \quad p \mid o(G/S)$

[$\because p^{\alpha} \mid o(G) \quad \dfrac{o(G)}{p^{\alpha}} = k$.

$o(G) = p^{\alpha} k$

$o(G/S) = \dfrac{o(G)}{o(S)} = \dfrac{k p^{\alpha}}{p^{\beta}} = k p^{\alpha - \beta}$

$p^n \mid o(G/S)$ where $n = \alpha - \beta$

$\Rightarrow p \mid o(G/S)$]

By Cauchy's them for abelian group, there exist a element $Sx \neq S$ in G/S such that $(Sx)^p = S = e$

Now, $((Sx)^p)^n = S$

Then $Sx^{p^n} = S$

$\Rightarrow x^{p^n} \in S$

$\Rightarrow (x^{p^n})^{o(S)} = e$

$\Rightarrow (x^{p^n})^{p^\beta} = e$

$\Rightarrow x^{p^{n+\beta}} = e$

$\Rightarrow x \in S$ [$\because n+\beta$ is an integer

which is a contradiction to $x \notin S$

[$\because Sx \neq S$].

$\beta = \alpha$

Hence S is a subgroup of order p^α

Hence the Theorem.

Corollary:

If G is abelian of order $o(G)$ and $p^\alpha | o(G)$, $p^{\alpha+1} \nmid o(G)$ then there is a unique subgroup of G of order p^α.

Proof:

Let G be an abelian group of order $o(G)$ and p is a prime such that $p^\alpha | o(G)$ and $p^{\alpha+1} \nmid o(G)$

$\therefore$ By Sylow's theorem for abelian groups, there is a subgroup S of G such that $o(S) = p^\alpha$

T.P uniqueness

Suppose there exist another subgroup T of G such that $o(T) = p^\alpha$ and $S \neq T$

Since G is abelian, $ST = TS$

$\therefore$ ST is a subgroup of G [By Lemma 2.5.1]

By Lagrange's Thm, $O(ST) \mid O(G)$

Now, $O(ST) = \dfrac{O(S)\,O(T)}{O(S\cap T)}$

Since $O(S) = p^{\alpha}$, $O(T) = p^{\alpha}$ and $O(S\cap T) < p^{\alpha}$ [$\because S\cap T$ is a subgroup of $S \& T$]

Let $O(S\cap T) = p^{\beta}$ for some $\beta < \alpha$

Then $O(ST) = \dfrac{p^{\alpha}p^{\alpha}}{p^{\beta}}$

$= p^{2\alpha-\beta}$

$= p^{\nu}$ where $\nu = 2\alpha - \beta > \alpha$

$\therefore p^{\nu} \mid O(G)$ and $\nu > \alpha$

which is a contradiction to $p^{\alpha+1} \nmid O(G)$

$\therefore$ There exist a unique subgroup of order p^{α} in G

Lemma: 2.7.2.

Let ϕ be a homomorphism of G onto $\bar{G}$ with kernal k for $\bar{H}$ a subgroup of $\bar{G}$, let H be defined by $H = \{x \in G \mid \phi(x) \in \bar{H}\}$. Then H is a subgroup of G and $H \supset K$; if $\bar{H}$ is normal in $\bar{G}$ then H is normal in G. Moreover this association sets up a 1-1 mapping from the set of all subgroups of $\bar{G}$ onto the set of all subgroups of G which contain k.

Proof:

Given $\phi : G \to \bar{G}$ is a onto homomorphism and $\bar{H}$ is a subgroup of $\bar{G}$ and

$H = \{x \in G \mid \phi(x) \in \bar{H}\}$

Claim H is a subgroup of G

Since $\phi(e) = \bar{e} \in \bar{H}$ [$\because \phi$ is homomorphism and $\bar{H}$ is a subgroup]

$\therefore e \in H$

$\therefore H \neq \phi$

Let $x, y \in H$

Then $\phi(x), \phi(y) \in \bar{H}$

Since $\bar{H}$ is a subgroup, $\phi(x)\phi(y) \in \bar{H}$

ie) $\phi(xy) \in \bar{H}$ [$\because \phi$ is a homomorphism]

ie) $xy \in H$ for all $x, y \in H$

Let $x \in H$

Then $\phi(x) \in \bar{H}$, $(\phi(x))^{-1} \in \bar{H}$

ie) $\phi(x^{-1}) \in \bar{H}$ [$\because \phi$ is a homomorphism]

$\therefore x^{-1} \in H \ \forall x \in H$

$\therefore H$ is a subgroup of G.

Clearly $K \subset H$

For Let $x \in K$.

Then $\phi(x) = \bar{e} \in \bar{H}$

$\therefore x \in H$.

Suppose $\bar{H}$ is normal in $\bar{G}$

Claim: H is normal in G

Let $g \in G$ and $h \in H$ for all $g \in G$ & $h \in H$

Then $\phi(g) \in \bar{G}$ and $\phi(h) \in \bar{H}$ (onto)

Since $\bar{H}$ is normal in $\bar{G}$,

$\phi(g)\phi(h)[\phi(g)]^{-1} \in \bar{H}$

ie) $\phi(ghg^{-1}) \in \bar{H}$ [$\because \phi$ is a homomorphism

$\Rightarrow ghg^{-1} \in H$

$\therefore H$ is a normal in G

We have $\varphi: G \to \bar{G}$ is a homomorphism

if we just consider the elements of H it induces the homomorphism of H onto $\bar{H}$ with kernal K.

By Fundamental thm for homomorphism,

$H/K \simeq \bar{H}$

Let L be a subgroup of G such that $K \subset L$

Let $\bar{L} = \{\bar{x} \in \bar{G} \,/\, \bar{x} = \varphi(l)$ for some $l \in L\}$

Claim: $\bar{L}$ is a subgroup of $\bar{G}$

Since $\bar{e} = \varphi(e)$, $e \in L$

$\therefore \bar{e} \in \bar{L} \neq \varphi$

Let $\bar{x}, \bar{y} \in \bar{L}$

Then $\varphi(x) = \bar{x}$ & $\varphi(y) = \bar{y}$ for some $x, y \in L$

Now, $\bar{x}\bar{y} = \varphi(x)\varphi(y)$

$= \varphi(xy)$, $xy \in L$ [$\because L$ is a subgroup]

$\therefore \bar{x}\bar{y} \in \bar{L}$

Let $\bar{x} \in \bar{L}$

Then $\bar{x} = \varphi(x)$, $x \in L$.

$\bar{x}^{-1} = [\varphi(x)]^{-1}$

$= \varphi(x^{-1})$, $x^{-1} \in L$ [$\because L$ is a subgroup]

$\therefore \bar{x}^{-1} \in \bar{L}$

$\therefore \bar{L}$ is a subgroup of $\bar{G}$

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Let $T = \{ y \in G \,/\, \varphi(y) \in \bar{L} \}$

Claim : $T = L$

Let $l \in L$

Then $\varphi(l) \in \bar{L}$

$\therefore l \in T$

Hence $L \subseteq T$

Let $t \in T$

Then $\varphi(t) \in \bar{L}$

$\Rightarrow \varphi(t) = \bar{l}$ for some $\bar{l} \in \bar{L}$

$\Rightarrow \varphi(t) = \varphi(l)$

$\Rightarrow \varphi(t)\varphi(l^{-1}) = \bar{e}$

$\Rightarrow \varphi(tl^{-1}) = \bar{e}$

$\Rightarrow tl^{-1} \in K \subset L$

$\Rightarrow tl^{-1} = l_1$ for some $l_1 \in L$.

$\Rightarrow t = l_1 l \in L$ [$\because L$ is a subgroup].

$\Rightarrow t \in L$.

Hence $T \subseteq L$

$\therefore T = L$

Thus we have set up a 1-1 correspondence between the set of all subgroups of $\bar{G}$ and the set of all subgroup of G containing K, Moreover in this correspondence a normal subgroup of G correspondce to a normal subgroup of $\bar{G}$

Thm 2.7.2

UQ Let φ be a homomorphism of G onto $\bar{G}$ with kernal K and let $\bar{N}$ be a normal subgroup of $\bar{G}$, $N = \{ x \in G \,/\, \varphi(x) \in \bar{N} \}$. Then $G/N \approx \bar{G}/\bar{N}$ ~~equiv~~

equivalently $G/N \approx \frac{\bar{G}/K}{\bar{N}/K}$

Proof:

Define $\psi : G \to \frac{\bar{G}}{\bar{N}}$ by $\psi(g) = \bar{N}\phi(g)$ for all $g \in G$

ψ is onto:

Let $\bar{N}\bar{g} \in \bar{G}/\bar{N}$; $\bar{g} \in \bar{G}$

Since $\phi : G \to \bar{G}$ is an onto homomorphism, there exist an element $g \in G$ such that $\phi(g) = \bar{g}$

$\therefore \psi(g) = \bar{N}\bar{g}$

$= \bar{N}\phi(g)$

ψ is a homomorphism:

Let $g, h \in G$

Then $\psi(gh) = \bar{N}\phi(gh)$

$= \bar{N}\phi(g)\phi(h)$ [$\because \phi$ is a homomorphism]

$= \bar{N}\phi(g)\bar{N}\phi(h)$

$= \psi(g)\psi(h)$

$\therefore$ By Fundamental thm of homomorphism,

$G/T \approx \frac{\bar{G}}{\bar{N}}$ where T is the Kernal of ψ

Claim. $T = N$

Let $n \in N$

Then $\phi(n) \in \bar{N}$

$\therefore \psi(n) = \bar{N}\phi(n) = \bar{N}$, [$\bar{N}$ is the indentity of $\frac{\bar{G}}{\bar{N}}$]

$\therefore n \in T$

Hence $N \subseteq T$

Let $t \in T$

Then $\varphi(t) = \overline{N}$

$\Rightarrow \overline{N} = \varphi(t)$

$\Rightarrow \overline{N} = \overline{N}\varphi(t)$

$\Rightarrow \varphi(t) \in \overline{N}$

$\Rightarrow t \in N$ (∵ by the defeniement of N)

Hence $T \subset N$

$\therefore T = N$

∴ By Fundamental thm of homomorphism,

$$\frac{G}{N} \approx \frac{\overline{G}}{\overline{N}} \quad \text{--- (1)}$$

Given φ ~~onto~~ from G onto $\overline{G}$ is a homomorphism with kernal K.

∴ By Fun thm of homo, $G/K \approx \overline{G}$ --- (2)

Now the restricted map $\varphi : N \to \overline{N}$ is an onto homomorphism with kernal K.

∴ By Fun thm of homo, $\frac{N}{K} \approx \overline{N}$ --- (3)

From (1), (2) & (3),

$$G/N \approx \frac{G/K}{N/K}$$